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THE ALGEBRAIC FRAMEWORK OF CERTAIN COMPLETE METABOLE GROUP ALGEBRAS

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Abstract: More precisely, G is a finite group defined as G=Z(G), which is isomorphic to the direct product of two cyclic groups with an order of p, where p is an odd prime. Additionally, Fq is a finite field that consists of q elements. The purpose of this inquiry is to determine the whole algebraic structure of the finite semi simple group algebra Fq[G].

1. INTRODUCTION

Let's assume that G is a finite group with an order that is relatively prime to q, and Fq is a finite field that includes q elements. The algebraic structure representing finite semi simple groups is often designated as Fq[G]. It is challenging to determine the whole algebraic structure of a semi simple group algebra, represented as Fq[G], when used in.

The primary difficulty in the field of group algebras is often defined by the existence of primitive central idempotents and the Wedderburn decomposition. It has significance in both theoretical mathematics and applied mathematics, particularly in the domain of coding theory.

Primitive central idempotents are essential for determining the smallest components of the Wedderburn decomposition of Fq[G]. Our main goal is to determine the whole algebraic structure of Fq[G] by identifying all the primitive central idempotents in its collection. Let us consider G as a group that may be expressed by the equation G=Z(G) \= Cp { Cp, where Cp is a cyclic group with a rank of p. Cornelissen et al. [2] have divided these classifications into nine separate groups. Out of the nine classes, only five consist of finite metabelian groups. Gupta et al. [3, 4] have found that the whole algebraic structure of Fq[G] has been calculated for G belonging to three of these classes.

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This study presents a comprehensive analysis of the algebraic structure of the finite semi simple group algebra Fq[G]. Here, G is a finite metabelian group belonging to one of the remaining classes.

$$G =< a, b, x, y | a^p = 1, b^p = y, x^{p^{m_1}} = y^{p^{m_2}} = 1, a^{-1}b^{-1}ab = x^{p^{m_1 - 1}}, x, y \in Z(G) >$$

2. NOTATIONS AND PRELIMINARIES

With jGj) equal to one and gcd(q) equal to one, let G be a finite group. Under the assumption that M, L, and G are subgroups, let us assume that M E L and L=M are cyclic of order n. A primitive nth root of unity in Fq is denoted by the symbol \, while the algebraic closure of Fq is denoted by the symbol Fq. The collection of q-cyclotomic variables is assumed to The generators of Irr(L=M) are included in the notion of cosets of Irr(L=M), where Irr(G) refers to the collection of irreducible characters present in G. Investigate the impact that T = NG(M)\NG(L) has on C(L=M) using the equation tC = t^1Ct; t 2 T; C 2 C(L=M). The collection of distinct orbits of C(L=M) under this action is denoted by R(L=M), while the stabilizer of any C 2 C(L=M) is denoted by EG(L=M). Take into consideration the fact that the total of the many conjugates of "C(L;M)" is eC(G; L;M).

$$\varepsilon_C(L, M) = |L|^{-1} \sum_{l \in L} tr_{\mathbb{F}_q(\zeta)/\mathbb{F}_q}(\chi(\bar{l}))l^{-1},\ C \in \mathcal{C}(L/M).$$

Consider the set of all subgroups D=K of AK=K that are cyclic. Additionally, consider AK=K to be the normal subgroup of G=K, which is an abelian subgroup of maximum order for K E G. Let { be the set of all subgroups D=K of AK=K. Let us assume that the collection of all equivalence class representatives for \ is denoted by the symbol \G=K. when the connection of conjugacy is in effect. Give out

$$\mathcal{S}_{G/\mathcal{K}} = \{(\mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \mid \mathcal{D}/\mathcal{K} \in \tau_{G/\mathcal{K}} \text{ and } \mathrm{Core}_G(\mathcal{D}) = \mathcal{K}\}$$

and

$$\mathcal{S} = \{(\mathcal{K}, \mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \mid \mathcal{K} \trianglelefteq G,\ \mathcal{S}_{G/\mathcal{K}} \neq \phi,\ (\mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \in \mathcal{S}_{G/\mathcal{K}}\}.$$

In accordance with the findings of Bakshi et al. [1], the structure of the finite semi simple metabelian group algebra is as follows:

Theorem 2.1. *[1, Theorem 2] Let* Fq *be a field containing* q *elements and* G *be a finite metabelian group such that* gcd(q; jGj) = 1. *Then*

$$\{e_C(G, A_{\mathcal{K}}, \mathcal{D}) \mid (\mathcal{K}, \mathcal{D}/\mathcal{K}, A_{\mathcal{K}}/\mathcal{K}) \in \mathcal{S},\ C \in \mathfrak{R}(A_{\mathcal{K}}/\mathcal{D})\}$$

is the complete set of primitive central idempotents of semi simple group algebra Fq[G].

The simple component corresponding to primitive central idempotent eC(G;AK;D) is Fq[G]eC(G;AK;D) =M[G:AK],Fqo(AK;D), the algebra of [G : AK] _ [G : AK] matrices over the field Fqo(AK;D) , where o(AK;D) =ord[AK:D](q)/[EG(AK=D):AK] and the number of simple components corresponding to eC(G;AK;D) is jR(AK=D)j.

3. ALGEBRAIC STRUCTURE OF Fq[G]

Let G be a group with the presentation:

$$G =< a, b, x, y | a^p = 1, b^p = y, x^{p^{m_1}} = y^{p^{m_2}} = 1, a^{-1}b^{-1}ab = x^{p^{m_1-1}}, \quad x, y \text{ central in } G > .$$

It can be easily seen that

$$G =< a, b, x | a^p = 1, b^{p^{m_2+1}} = 1, x^{p^{m_1}} = 1, a^{-1}b^{-1}ab = x^{p^{m_1-1}}, \quad x, b^p \text{ central in } G > .$$

Theorem 3.1. Let m1;m2 > 1. For m1 _ m2, the complete algebraic structure of semi simple group algebra Fq[G], G as defined above, is given as follows:

Primitive Central Idempotents for m1 _ m2**.**

$$e_C(G, G, < x, a, b >),\ C \in \mathfrak{R}(G/ < x, a, b >);$$
$$e_C(G, G, < x, a >),\ C \in \mathfrak{R}(G/ < x, a >);$$
$$e_C(G, G, < x, b >),\ C \in \mathfrak{R}(G/ < x, b >);$$
$$e_C(G, G, < x, a, b^{p^j} >),\ C \in \mathfrak{R}(G/ < x, a, b^{p^j} >)\ 1 \le j \le m_2;$$
$$e_C(G, G, < x, a^i b^{p^j} >),\ C \in \mathfrak{R}(G/ < x, a^i b^{p^j} >),\ 1 \le i \le p-1,\ 0 \le j \le m_2;$$
$$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, b >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, b >),\quad 0 \le i \le p-1,\ 1 \le v \le m_1 - 1;$$
$$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >),\quad 1 \le j \le m_2 + 1 - v,\ ,0 \le i \le p-1,\ 1 \le v \le m_1 - 1,\ \gcd(k, p^v) = 1;$$
$$e_C(G, G, < x^{p^v}, x^{ip^{v-1}}a, x^k b >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}}a, x^k b >),$$

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$$
\begin{aligned}
&\quad 0 \le i \le p-1,\ \gcd(k, p^v) = p^\alpha,\ 0 \le \alpha \le m_1 - 1;\\
&e_C(G, <b, x>, <b>),\ \ C \in \mathfrak{R}(<b, x> / <b>);\\
&e_C(G, <a, x, y>, <a, x^j y>),\ \ C \in \mathfrak{R}(<a, x, y> / <a, x^j y>),\\
&\quad \gcd(j, p^{m_1}) = p^\alpha,\ 0 \le \alpha \le m_1 - 1;\\
&e_C(G, <a, x, y>, <a, x^j y^{p^\beta}>),\ \ C \in \mathfrak{R}(<a, x, y> / <a, x^j y^{p^\beta}>),\\
&\quad \gcd(j, p^{m_1}) = 1,\ 1 \le \beta \le m_2 - m_1.
\end{aligned}
$$

Wedderburn Decomposition for m1 _ m2**.**

$$
\begin{aligned}
\mathbb{F}_q[G] \cong & \mathbb{F}_q \oplus \left(\mathbb{F}_{q^{f_1}}\right)^{\frac{p-1}{f_1}} \oplus \left(\mathbb{F}_{q^{f_{m_2+1}}}\right)^{\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}} \oplus_{j=1}^{m_2} \left(\mathbb{F}_{q^{f_j}}\right)^{\frac{p^j-p^{j-1}}{f_j}}\\
& \oplus_{j=0}^{m_2} \left(\mathbb{F}_{q^{f_{j+1}}}\right)^{\frac{p^j(p-1)^2}{f_{j+1}}} \oplus_{v=1}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{v+1}-p^v}{f_v}}\\
& \oplus_{v=1}^{m_1-1} \oplus_{j=1}^{m_2+1-v} \left(\mathbb{F}_{q^{f_{j+v}}}\right)^{\frac{p^{2v+j-1}(p-1)^2}{f_{j+v}}} \oplus_{v=1}^{m_1-1} \oplus_{\alpha=0}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{2v-\alpha-1}(p-1)^2}{f_v}}\\
& \oplus M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}} \oplus_{\alpha=0}^{m_1-1} M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{2m_1-\alpha-2}(p-1)^2}{f_{m_1}}}\\
& \oplus_{\beta=1}^{m_2-m_1} M_p\left(\mathbb{F}_{q^{f_{m_1+\beta}}}\right)^{\frac{p^{2m_1+\beta-2}(p-1)^2}{f_{m_1+\beta}}}
\end{aligned}
$$

Evidence. First, we shall identify every normal subgroup of G.

Let $\mathcal{K} \trianglelefteq G$ such that $\mathcal{K} \cap <x> \neq \{e\}$, then $\mathcal{K} \cap <x> = <x^{p^v}>, 0 \le v \le m_1 - 1$. For $\mathcal{K} \cap <x> = <x>$, it can be easily seen that $\mathcal{K}$ is either $<x>$ or $<x, a>$ or $<x, b^{p^j}>$ or $<x, a, b^{p^j}>$ or $<x, a^i b^{p^j}>, 1 \le i \le p-1, 0 \le j \le m_2$.

Assume that $\mathcal{K} \cap <x> = <x^{p^v}>$, $1 \le v \le m_1 - 1$. Now $\mathcal{K}/<x^{p^v}>$ is isomorphic to one of the following $<x>$, $<a<x>>$, $<b^{p^j}<x>>$, $<a^i b^{p^j}<x>>$, $<a<x>, b^{p^j}<x>>$, $1 \le i \le p-1$, $0 \le j \le m_2$. Let $\mathcal{K}/<x^{p^v}> \cong <x>$, then $\mathcal{K} = <x^{p^v}>$. Let $\mathcal{K}/<x^{p^v}> \cong <a<x>>$, then $\mathcal{K} = <x^{p^v}, x^i a>$, which implies for $i = p^v$, $\mathcal{K} = <x^{p^v}, a>$ and if $\gcd(i, p^v) = p^\alpha$, then $(x^{p^\alpha} a)^p = x^{p^{\alpha+1}} \in \mathcal{K}$ if and only if $\alpha + 1 \ge v$, i.e., $\alpha \ge v - 1$. Hence in this case $\mathcal{K} = <x^{p^v}, a>$ or $<x^{p^v}, x^{ip^{v-1}} a>, 1 \le i \le p-1$.

Let $\mathcal{K}/<x^{p^v}> \cong <b^{p^j}<x>>$, $0 \le j \le m_2$. Then $\mathcal{K} = <x^{p^v}, x^k b^{p^j}>$. If $k = p^v$, then $\mathcal{K} = <x^{p^v}, b^{p^j}>$ and if $k < p^v$, then $\mathcal{K} = <x^{p^v}, x^k b^{p^j}>$, $\gcd(k, p^v) = p^\alpha$, $0 \le \alpha \le m_1 - 1$. Consider $(x^{p^\alpha} b^{p^j})^{p^{m_2+1-j}} = x^{p^{\alpha+m_2+1-j}} \in \mathcal{K}$ if and only if $\alpha + m_2 + 1 - j \ge v$, i.e., $j \le \alpha + m_2 + 1 - v$. Thus in this case $\mathcal{K} = <x^{p^v}, x^k b^{p^j}>$, $\gcd(k, p^v) = p^\alpha, 0 \le \alpha \le m_1 - 1, 0 \le j \le \alpha + m_2 + 1 - v$. Further if $j = 0$, then $(x^{p^\alpha} b)^{p^{m_2+1}} = x^{p^{\alpha+m_2+1}} \in \mathcal{K}$ if, and only if, $\alpha + m_2 + 1 \ge v$, i.e., $v - m_2 - 1 \le \alpha$. Thus

$\mathcal{K} = <x^{p^v}, b>$ or $<x^{p^v}, x^k b>$, $\gcd(k, p^v) = p^\alpha$, $\max\{0, v - m_2 - 1\} \le \alpha \le m_1 - 1$. If $1 \le j \le m_2 + 1 - v + \alpha$, then $\mathcal{K} = <x^{p^v}, b^{p^j}>$ or $<x^{p^v}, x^k b^{p^j}>$, $\gcd(k, p^v) = 1$, $1 \le j \le m_2 + 1 - v$.

By following the same procedure in all the above cases, we will get that the normal subgroups K such that K \ G 6= feg, are as follows:

$$
\begin{aligned}
&< x >,\ < x, a >,\ < x, b^{p^j} >,\ < x, a^i b^{p^j} >,\ < x, a, b^{p^j} >,\\
&\quad 1 \le i \le p-1,\ 0 \le j \le m_2,\\
&< x^{p^v}, a >,\ < x^{p^v}, x^{ip^{v-1}} a >,\ 1 \le i \le p-1,\ 1 \le v \le m_1 - 1,\\
&< x^{p^v}, b >,\ < x^{p^v}, x^k b >,\ \gcd(k, p^v) = p^\alpha,\ \max\{0, v - m_2 - 1\} \le \alpha \le m_1 - 1,\\
&\quad 1 \le v \le m_1 - 1,\\
&< x^{p^v}, b^{p^j} >,\ < x^{p^v}, x^k b^{p^j} >,\ \gcd(k, p^v) = 1,\ 1 \le j \le m_2 + 1 - v,\\
&< x^{p^v}, a^i b^{p^j} >,\ < x^{p^v}, x^k a^i b^{p^j} >,\ \gcd(k, p^v) = p^\alpha,\ 0 \le \alpha \le m_1 - 1,\\
&\quad 1 \le v \le m_1 - 1\ 0 \le j \le m_2 + 1 - v + \alpha,\\
&< x^{p^v}, x^{ip^{v-1}} a, b^{p^j} >,\ 0 \le i \le p-1,\ 0 \le j \le m_2,\ 1 \le v \le m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b >,\ 0 \le i \le p-1,\ \gcd(k, p^v) = p^\alpha,\ 1 \le v \le m_1 - 1\\
&\quad \max\{0, v - m_2 - 1\} \le \alpha \le m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j} >,\ 0 \le i \le p-1,\ \gcd(k, p^v) = 1,\\
&\quad 1 \le j \le m_2 + 1 - v,\ 1 \le v \le m_1 - 1.
\end{aligned}
$$

Observe that if K\G =< xpv >; 0 _ v _ m1   1, then G0 _ K and hence G=K

is abelian. Thus

$$
S_{G/\mathcal{K}} = \begin{cases} (< 1 >, G/\mathcal{K}), & \text{if } G/\mathcal{K} \text{ is cyclic,} \\ \phi & \text{otherwise.} \end{cases}
$$

Out of these normal subgroups following have cyclic quotient with G:

$$
\begin{aligned}
&< x, a >,\ < x, b >,\ < x, a, b >,\ < x, a, b^{p^j} >,\ 1 \le j \le m_2,\\
&< x, a^i b^{p^j} >,\ 1 \le i \le p-1,\ 0 \le j \le m_2,\\
&< x^{p^v}, x^{ip^{v-1}} a, b >,\ 0 \le i, \le p-1,\ 1 \le v \le m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b >,\ 1 \le v \le m_1 - 1,\ 0 \le i \le p-1,\\
&\quad \gcd(k, p^v) = p^\alpha,\ \max\{0, v - m_2 - 1\} \le \alpha \le m_1 - 1,\\
&< x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j} >,\ 1 \le v \le m_1 - 1,\ 0 \le i \le p-1,\\
&\quad \gcd(k, p^v) = 1,\ 0 \le j \le m_2 + 1 - v.
\end{aligned}
$$

Now, assume that $\mathcal{K} \cap G = \{e\}$, then $\mathcal{K} = \{e\}$ or $< y^{p^j} >$, $0 \le j \le p-1$ or $< x^j y^{p^\beta} >$, $\gcd(j, p^{m_1}) = p^\alpha$, $0 \le \alpha \le m_1 - 1$, $0 \le \beta \le m_2 - m_1$.

If $\mathcal{K} = \{e\}$ or $< y^{p^j} >$, $1 \le j \le p-1$, then $S_{G/\mathcal{K}} = \phi$.

If $\mathcal{K} = \{y\}$, then $S_{G/\mathcal{K}} = \{< b > /\mathcal{K}, < b, x > /\mathcal{K}\}$.

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If $\mathcal{K} =< x^j y >$, $\gcd(j, p^{m_1}) = p^\alpha$, $0 \le \alpha \le m_1 - 1$, then $S_{G/\mathcal{K}} = \{< a, x^j y > /\mathcal{K}, < a, x, y > /\mathcal{K}\}$.

If $\mathcal{K} =< x^j y^{p^\beta} >$, $\gcd(j, p^{m_1}) = 1$, $1 \le \beta \le m_2 - m_1$, then $S_{G/\mathcal{K}} = \{< a, x^j y^{p^\beta} > /\mathcal{K}, < a, x, y > /\mathcal{K}\}$.

Corresponding to these normal subgroups, o(AK;D) and jR(A;K ;D)j have been given in Table 1.

Table 2

$\mathcal{K}$	$(D, A_{\mathcal{K}})$	$o(A_{\mathcal{K}}, D)$	$\lvert\mathfrak{R}(A_{\mathcal{K}}/D)\rvert$
$< x, a, b >$	(G, G)	1	1
$< x, a >$	$(\mathcal{K}, G)$	f_{m_2+1}	$\frac{p^{m_2+1} - p^{m_2}}{f_{m_2+1}}$
$< x, b >$	$(\mathcal{K}, G)$	f_1	$\frac{p-1}{f_1}$
$< x, a, b^{p^j} >$	$(\mathcal{K}, G)$	f_j	$\frac{p^j - p^{j-1}}{f_j}$

Thus by using this table primitive central idempotents and Wedderburn decomposition given in Theorem 3.2 can be easily obtained.

We have obtained the complete algebraic structure of Fq[G] for m1;m2 > 1.

In the following theorems, we have obtained the same for cases m1 _ 1;m2 = 1 and m1 = 1;m2 _ 1.

Theorem 3.3. *Let* G *be a group defined above. For* m1 _ 1;m2 = 1 *the complete algebraic structure of semisimple group algebra* Fq[G] *is given as follows:*

$\mathcal{K}$	$(D, A_{\mathcal{K}})$	$o(A_{\mathcal{K}}, D)$	$\lvert\mathfrak{R}(A_{\mathcal{K}}/D)\rvert$
$< x, a, b >$	(G, G)	1	1
$< x, a >$	$(\mathcal{K}, G)$	f_{m_2+1}	$\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}$
$< x, b >$	$(\mathcal{K}, G)$	f_1	$\frac{p-1}{f_1}$
$< x, a, b^{p^j} >$	$(\mathcal{K}, G)$	f_j	$\frac{p^j-p^{j-1}}{f_j}$
$< x, a^i b^{p^j} >, 1 \leq i \leq p-1$ $0 \leq j \leq m_2$	$(\mathcal{K}, G)$	f_{j+1}	$\frac{p^{j+1}-p^j}{f_{j+1}}$
$< x^{p^v}, x^{ip^{v-1}}a, b >, 1 \leq v \leq m_1 - 1$ $0 \leq i \leq p-1$	$(\mathcal{K}, G)$	f_v	$\frac{p^v-p^{v-1})}{f_v}$
$< x^{p^v}, x^{ip^{v-1}}a, x^k b >,$ $0 \leq i \leq p-1,$ $1 \leq v \leq m_1 - 1, \gcd(k, p^v) = p^\alpha,$ $\max\{0, v - m_2 - 1\} \leq \alpha \leq m_1 - 1$	$(\mathcal{K}, G)$	f_v	$\frac{p^v-p^{v-1}}{f_v}$
$< x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >,$ $0 \leq i \leq p-1,$ $1 \leq v \leq m_1 - 1, \gcd(k, p^v) = 1$ $1 \leq j \leq m_2 + 1 - v$	$(\mathcal{K}, G)$	f_{j+v}	$\frac{p^{j+v}-p^{j+v-1}}{f_{j+v}}$
$< y >$	$(< b >, < b, x >)$	f_{m_1}	$\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}$
$< x^j y >$ $\gcd(j, p^{m_1}) = p^\alpha, 0 \leq \alpha \leq m_1 - 1$	$(< a, x^j y >,$ $< a, x, y >)$	f_{m_1}	$\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}$
$< x^j y^{p^\beta} >$ $\gcd(j, p^{m_1}) = 1, 1 \leq \beta \leq m_2 - m_1$	$(< a, x^j y^{p^\beta} >,$ $< a, x, y >)$	$f_{m_1+\beta}$	$\frac{p^{m_1+\beta}-p^{m_1+\beta-1}}{f_{m_1+\beta}}$

Thus by using this table primitive central idempotents and Wedderburn decomposition given in Theorem 3.1 can be easily obtained

Theorem 3.2. *Let m1;m2 > 1. For m1 > m2, the complete algebraic structure of semisimple group algebra Fq[G], G as defined above, is given as follows:*

***Primitive Central Idempotents for** m1 > m2.*

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$$
\begin{aligned}
&e_C(G, G, <x, a, b>),\ C \in \Re(G/<x, a, b>);\\
&e_C(G, G, <x, a>),\ C \in \Re(G/<x, a>);\\
&e_C(G, G, <x, b>),\ C \in \Re(G/<x, b>);\\
&e_C(G, G, <x, a, b^{p^j}>),\ C \in \Re(G/<x, a, b^{p^j}>)\ 1 \leq j \leq m_2;\\
&e_C(G, G, <x, a^i b^{p^j}>),\ C \in \Re(G/<x, a^i b^{p^j}>),\ 1 \leq i \leq p-1,\ 0 \leq j \leq m_2;\\
&e_C(G, G, <x^{p^v}, x^{ip^{v-1}} a, b>),\ C \in \Re(G/<x^{p^v}, x^{ip^{v-1}} a, b>),\\
&\quad 0 \leq i \leq p-1,\ 1 \leq v \leq m_1-1;\\
&e_C(G, G, <x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j}>),\ C \in \Re(G/<x^{p^v}, x^{ip^{v-1}} a, x^k b^{p^j}>),\\
&\quad 1 \leq j \leq m_2+1-v,\ ,0 \leq i \leq p-1,\ 1 \leq v \leq m_2,\ \gcd(k, p^v) = 1;\\
&e_C(G, G, <x^{p^v}, x^{ip^{v-1}} a, x^k b>),\ C \in \Re(G/<x^{p^v}, x^{ip^{v-1}} a, x^k b>),\\
&\quad 0 \leq i \leq p-1,\ \gcd(k, p^v) = p^\alpha,\ \max\{0, v-m_2-1\} \leq \alpha \leq v-1;\\
&e_C(G, <b, x>, <b>),\ C \in \Re(<b, x>/<b>);\\
&e_C(G, <a, x, y>, <a, x^j y>),\ C \in \Re(<a, x, y>/<a, x^j y>),\\
&\quad \gcd(j, p^{m_1}) = p^\alpha,\ m_1 - m_2 \leq \alpha \leq m_1 - 1.
\end{aligned}
$$

Wedderburn Decomposition for m1 > m2**.**

$$
\begin{aligned}
\mathbb{F}_q[G] \cong\ & \mathbb{F}_q \oplus \left(\mathbb{F}_{q^{f_1}}\right)^{\frac{p-1}{f_1}} \oplus \left(\mathbb{F}_{q^{f_{m_2+1}}}\right)^{\frac{p^{m_2+1}-p^{m_2}}{f_{m_2+1}}} \oplus_{j=1}^{m_2} \left(\mathbb{F}_{q^{f_j}}\right)^{\frac{p^j-p^{j-1}}{f_j}}\\
& \oplus_{j=0}^{m_2} \left(\mathbb{F}_{q^{f_{j+1}}}\right)^{\frac{p^j(p-1)^2}{f_{j+1}}} \oplus_{v=1}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{v+1}-p^v}{f_v}}\\
& \oplus_{v=1}^{m_2} \oplus_{j=1}^{m_2+1-v} \left(\mathbb{F}_{q^{f_{j+v}}}\right)^{\frac{p^{2v+j-1}(p-1)^2}{f_{j+v}}} \oplus_{v=1}^{m_2} \oplus_{\alpha=0}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{2v-\alpha-1}(p-1)^2}{f_v}}\\
& \oplus_{v=m_2+1}^{m_1-1} \oplus_{\alpha=v-m_2-1}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\frac{p^{2v-\alpha-1}(p-1)^2}{f_v}} \oplus M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{m_1}-p^{m_1-1}}{f_{m_1}}}\\
& \oplus_{\alpha=m_1-m_2}^{m_1-1} M_p\left(\mathbb{F}_{q^{f_{m_1}}}\right)^{\frac{p^{2m_1-\alpha-2}(p-1)^2}{f_{m_1}}}
\end{aligned}
$$

Proof. By following the same procedure as in Theorem 3.1, we will get that the normal subgroups K such that K \ G 6= feg, are as follows:

$$
\begin{aligned}
&<x>, <x, a>, <x, b^{p^j}>, <x, a^i b^{p^j}>, <x, a, b^{p^j}>,\\
&\quad 1 \leq i \leq p-1,\ 0 \leq j \leq m_2,\\
&<x^{p^v}, a>, <x^{p^v}, x^{ip^{v-1}} a>, 1 \leq i \leq p-1,\ 1 \leq v \leq m_1-1,\\
&<x^{p^v}, b>, <x^{p^v}, x^k b>, \gcd(k, p^v) = p^\alpha,
\end{aligned}
$$

$$\max\{0, v-m_2-1\} \leq \alpha \leq v-1,\ 1 \leq v \leq m_1-1,$$
$$< x^{p^v}, b^{p^j} >, < x^{p^v}, x^k b^{p^j} >,\ \gcd(k,p^v)=1,\ 1 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_2,$$
$$< x^{p^v}, a^i b^j >, < x^{p^v}, x^k a^i b^j >, \gcd(k,p^v)=p^\alpha, \max\{0, v-m_2-1\} \leq \alpha \leq v-1,$$
$$< x^{p^v}, a^i b^{p^j} >, < x^{p^v}, x^k a^i b^{p^j} >,\ \gcd(k,p^v)=1,$$
$$1 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_2,$$
$$< x^{p^v}, x^{ip^{v-1}}a, b^{p^j} >,\ 0 \leq i \leq p-1,\ 0 \leq j \leq m_2,\ 1 \leq v \leq m_1-1,$$
$$< x^{p^v}, x^{ip^{v-1}}a, x^k b >,\ 0 \leq i \leq p-1,\ \gcd(k,p^v)=p^\alpha,$$
$$\max\{0, v-m_2-1\} \leq \alpha \leq v-1,\ 1 \leq v \leq m_1-1,$$
$$< x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >,\ 0 \leq i \leq p-1,\ \gcd(k,p^v)=1,$$
$$1 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_2.$$

Out of the normal subgroups listed above, only following have cyclic quotient

with G:

$$< x, a >, < x, b >, < x, a, b >, < x, a, b^{p^j} >,\ 1 \leq j \leq m_2,$$
$$< x, a^i b^{p^j} >,\ 1 \leq i \leq p-1,\ 0 \leq j \leq m_2,$$
$$< x^{p^v}, x^{ip^{v-1}}a, b >,\ 0 \leq i, \leq p-1,\ 1 \leq v \leq m_1-1,$$
$$< x^{p^v}, x^{ip^{v-1}}a, x^k b >,\ 1 \leq v \leq m_1-1,\ 0 \leq i \leq p-1,\ \gcd(k,p^v)=p^\alpha,$$
$$\max\{0, v-m_2-1\} \leq \alpha \leq v-1,$$
$$< x^{p^v}, x^{ip^{v-1}}a, x^k b^{p^j} >,\ 0 \leq i \leq p-1,\ \gcd(k,p^v)=1,$$
$$0 \leq j \leq m_2+1-v,\ 1 \leq v \leq m_1-1.$$

Now, assume that $\mathcal{K} \cap G = \{e\}$. In this case, if $\mathcal{K} = \{e\}$ or $< y^{p^j} >$, $1 \leq j \leq p-1$, then $S_{G/\mathcal{K}} = \phi$; if $\mathcal{K} = \{y\}$, then $S_{G/\mathcal{K}} = \{< b > /\mathcal{K}, < b, x > /\mathcal{K}\}$; if $\mathcal{K} =< x^j y >$, $\gcd(j, p^{m_1}) = p^\alpha$, $m_1 - m_2 \leq \alpha \leq m_1 - 1$; then $S_{G/\mathcal{K}} = \{< a, x^j y > /\mathcal{K}, < a, x, y > /\mathcal{K}\}$.

Corresponding to these normal subgroups, o(AK;D) and jR(A;K ;D)j have

been given in Table 2.

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$$e_C(G, G, < x, a, b >),\ C \in \mathfrak{R}(G/ < x, a, b >);$$
$$e_C(G, G, < x, a >),\ C \in \mathfrak{R}(G/ < x, a >);$$
$$e_C(G, G, < x, b >),\ C \in \mathfrak{R}(G/ < x, b >);$$
$$e_C(G, G, < x, a, b^p >),\ C \in \mathfrak{R}(G/ < x, a, b^p >);$$
$$e_C(G, G, < x, a^i b >),\ C \in \mathfrak{R}(G/ < x, a^i b >),\ 1 \leq i \leq p-1;$$
$$e_C(G, G, < x, a^i b^p >),\ C \in \mathfrak{R}(G/ < x, a^i b^p >),\ 1 \leq i \leq p-1;$$
$$e_C(G, G, < x^p, a, x^i b^p >),\ C \in \mathfrak{R}(G/ < x^p, a, x^i b^p >),\ 1 \leq i \leq p-1;$$
$$e_C(G, G, < x^p, x^i a, x^j b^p >),\ C \in \mathfrak{R}(G/ < x^p, x^i a, x^j b^p >),\ 1 \leq i, j \leq p-1;$$
$$e_C(G, G, < x^{p^v}, x^{ip^{v-1}} a, x^j b^p >),\ C \in \mathfrak{R}(G/ < x^{p^v}, x^{ip^{v-1}} a, x^{jp^{v-1}} b >),$$
$$0 \leq i, j \leq p-1,\ 1 \leq v \leq m_1 - 1;$$

CONCLUSION

The algebraic framework of complete metabole group algebras offers a rich and robust area of study, with numerous implications for both theoretical and practical applications. The insights gained from this research not only enhance our understanding of these algebras but also pave the way for further investigations into more complex algebraic systems. Future research could focus on extending these findings to other types of group algebras and exploring their interactions with different mathematical structures.

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